A UNIQUE PRIME DECOMPOSITION RESULT FOR WREATH PRODUCT FACTORS

J. OWEN SIZEMORE AND ADAM WINCHESTER

Abstract. We use malleable deformations combined with spectral gap rigidity theory, in the framework of Popa's deformation/rigidity theory to prove unique tensor product decomposition results for II_1 factors arising as tensor product of wreath product factors. We also obtain a similar result regarding measure equivalence decomposition of direct products of such groups.

Contents

Introduction

A major goal of the study of II_1 factors is the classification of these algebras based on the "input data" that goes into their construction. A significant landmark was the result, due to Connes [Co76], that all amenable II_1 factors are isomorphic. However, in the non-amenable realm there is a much greater variety, and a striking classification theory has developed.

One thrust of this research is to determine if some algebra which, *a priori*, is constructed in one manner, can be obtained in some other manner. For example, if we have a II_1 factor that we know to be a free product of two II_1 factors, is it also possible to be the tensor product of two (possibly different) II_1 factors?

In this vein we study whether certain factors can be written as a tensor product in two distinct ways. Such results go back to the study of prime factors, (ie. a factor which cannot be written as the tensor product of two other II_1 factors.) The first result was obtained by Popa in, [Po83], where he showed that the group von Neumann algebra of an uncountable free group is prime.

Later, in [Ge98], Ge proves that all group factors coming from finitely generated free groups are prime. Using C^* techniques this was greatly generalized by Ozawa, [Oz03], to show that all i.c.c. Gromov hyperbolic groups give rise to prime factors. Also, using his deformation/rigidity theory, Popa showed in [Po06a] that all II_1 factors arising from the Bernoulli actions of nonamenable groups are prime. Further, Peterson used his derivation approach to deformation/rigidity ([Pe06]) to prove that

Date: December 14, 2011.

any II_1 factor coming from a countable group with positive first l^2-betti number is also prime. Finally we should also note that using Popa's deformation/rigidity theory, Chifan and Houdayer, [CH08], gave many more examples of prime II_1-factors coming from amalgamated free products.

A natural question about prime factors is whether a tensor product of a finite number of such factors $P_1, P_2, ..., P_n$, has a "unique prime factor decomposition", i.e., if $P_1\overline{\otimes}...\overline{\otimes}P_n = Q_1\overline{\otimes}....\overline{\otimes}Q_m$, for some other prime factors Q_j, forces $n = m$ and P_i unitary conjugate to Q_i, modulo some permutation of indices and modulo some "rescaling" by appropriate amplifications of the prime factors involved. A first such result was obtained by Ozawa and Popa in [OP03], where a combination of C^* techniques from [Oz03] and intertwining techniques from [Po03] is used to show that any II_1 factor arising from a tensor product of hyperbolic group factors has such a unique tensor product decomposition.

In this paper we prove an analogous unique prime factor decomposition result for tensor products of wreath product II_1 factors. More precisely, we prove the following result:

Theorem 0.1. *Let $A_1, \ldots, A_n$ be non-trivial amenable groups; $H_1, \ldots, H_n$ be non-amenable groups; and $Q_1, \ldots, Q_k$ be diffuse von Neumann algebras such that*

$$M = L(A_1 \wr H_1)\overline{\otimes}\ldots\overline{\otimes}L(A_n \wr H_n) = Q_1\overline{\otimes}\ldots\overline{\otimes}Q_k$$

If $k \geq n$, then $n = k$, and after permutation of indices we have that $L(A_i \wr H_i) \simeq Q_i^{t_i}$ for some positive numbers $t_1, t_2, \ldots t_n$ whose product is 1.

Also we have a natural generalization of this theorem to unique measure-equivalence decomposition results of finite products of wreath product groups. Such results were achieved for products of groups of the class $\mathcal{C}_{reg}$ by Monod and Shalom (Theorem 1.16 in [MS06]), for products of bi-exact groups by Sako (Theorem 4 in [Sa09]), and for products of groups in $\mathcal{QH}_{reg}$ by Chifan and Sinclair (Corollary C in [CS10].)

Corollary 0.2. *Let $A_1, \ldots, A_n$ be non-trivial amenable groups; $H_1, \ldots, H_n$ be non-amenable groups; and $K_1, \ldots, K_m$ be groups such that*

$$A_1 \wr H_1 \times \cdots \times A_n \wr H_n \simeq_{ME} K1 \times \cdots \times K_m$$

If $m \geq n$, then $n = m$, and after permutation of indices we have that $A_i \wr H_i \simeq_{ME} K_i$.

We prove these results by using deformation/rigidity theory. More precisely, we use the malleable deformation for wreath product group factors in [CPS11], combined with Popa's spectral gap rigidity and intertwining by bimodules techniques.

Acknowledgements: We would like to thank Sorin Popa for suggesting this problem and his encouragement throughout.

1. Preliminaries

Intertwining by Bimodules: Let us recall Popa's intertwining by bimodules technique. This is a crucial tool for locating subalgebras of II_1-factors, and is summed up in the following theorem:

Theorem 1.1 (Popa, [Po03]). *Let $P, Q \subset M$ be finite von Neumann algebras. Then the following are equivalent:*

(1) *There exists nonzero projections $p \in P, q \in Q$, a nonzero partial isometry $v \in M$, and a *-homomorphism $\varphi : pPp \to qQq$ such that $vx = \varphi(x)v, \forall x \in pPp$*
(2) *There is a sub-$P-Q$-bimodule $\mathcal{H} \subset L^2(M)$ that is finitely generated as a right Q-module.*
(3) *There is no sequence $u_n \in \mathcal{U}(P)$ such that*

$$lim_{n\to\infty} \|E_Q(xu_ny)\|_2 \to 0, \forall x, y \in M$$

If any of the above conditions hold we say that *a corner of P embedds in Q inside M*, denoted $P \prec_M Q$.

Following [OP07] we have the following definition:

Definition 1.2. Let $P, Q \subset M$ be finite von Neuman algebras. We say that P *is amenable over Q inside M*, which we denote $P \lessdot_M Q$, if there is a P-central state, φ, on $\langle M, e_Q\rangle$ such that $\varphi|_M = \tau$, where τ is the trace on M.

Let us note that by Theorem 2.1 in [OP07] $P \lessdot_M Q$ is equivalent to $L^2(P) \prec \bigoplus L^2(\langle M, e_Q\rangle)$ as P-bimodules. Further, if $P \prec_M Q$ then $L^2(M)$ contains a sub P-Q-module, $\mathcal{H}$, that is finitely generated as a right Q module. Therefore, the projection onto this module will commute with the right action of Q and will have finite trace. Therefore, it will be a vector in $L^2(\langle M, e_N\rangle)$. Further, it will also commute with P, so if we look at $L^2(\langle M, e_N\rangle)$ as a P-bimodule, it will contain a central vector. Since strong containment implies weak containment we get the following observation.

Proposition 1.3. *Let $P, Q \subset M$ be von Neumann algebras. If $P \prec_M Q$ then $P \lessdot_M Q$.*

Deformation of Wreath Products: Let A and H be discrete groups. Then following standard notation we let $A \wr H = A^H \rtimes H$ denote the standard wreath product. Throughout this paper we will consider trace preserving actions of $A \wr H$ on a finite von Neuman algebra N, and we consider the resulting crossed product algebra $M = N \rtimes A \wr H$.

Let $\tilde{A} = A * \mathbb{Z}$. If we let $u \in L(\tilde{A})$ denote the Haar unitary that generates $L(\mathbb{Z})$ then for every $t \in \mathbb{R}$, we define $u^t \doteq \exp(ith) \in L\mathbb{Z}$. This allows us to define $\theta_t \in \mathrm{Aut}(L(\tilde{A}))$ by $\theta_t(x) = u^t x (u^*)^t$. By applying this automorphism in each coordinate we can get an automorphism of $L(\tilde{A}^H)$. Since the action of H is by permuting the coordinates, it commutes with θ_t and so we can extend it to $L(\tilde{A} \wr H)$. If we now declare that the Haar unitaries in each coordinate do not act on the algebra N, then we can extend to an automorphism, which we still denote by θ_t of $\tilde{M} = N \rtimes \tilde{A} \wr H$.

I is easy to see that $\lim_{t\to 0} \|u^t - 1\|_2 = 0$ and hence we have $\lim_{t\to 0} \|\theta_t(x) - x\|_2 = 0$ for all $x \in \tilde{M}$. Therefore, the path $(\theta_t)_{t\in\mathbb{R}}$ is a deformation by automorphisms of $\tilde{M}$.

Next we show that θ_t admits a "symmetry", i.e. there exists an automorphism β of $\tilde{M}$ satisfying the following relations:

$$\beta^2 = 1,\ \beta_{|_M} = id_{|_M},\ \beta\theta_t\beta = \theta_{-t},\ \text{for all } t \in \mathbb{R}.$$

To see this, first define $\beta_{|_{LA^I}} = id_{|_{LA^I}}$ and then for every $h \in H$ we let $(u)_h$ be the element in $L\tilde{A}^H$ whose h^{th}-entry is u and 1 otherwise. On elements of this form we define $\beta((u)_h) = (u^*)_h$, and since β commutes with the actions of H on A^H, it extends to an automorphism of $L(\tilde{A} \wr H)$ by acting identically on $L(H)$. Finally, the automorphism β extends to an automorphism of $\tilde{M}$, still denoted by β, which acts trivially on A.

Let us note that, with this choice of β, θ_t is an *s-malleable deformation* of $\tilde{M}$ in the sense of [Po03]. In fact, this is the same deformation that the first author used in [CPS11], and is inspired by similar free malleable deformations in [Po01, IPP05, Io06], so we refer to this previous work for additional discussion.

2. Intertwining Techniques for Wreath Products

In this section we prove the necessary intertwining results for II_1 factors arising from wreath product groups that we will need in order to prove our desired uniqueness of of tensor product decomposition.

The following proposition is a relative version of Lemma 4.2 in [CPS11], and will follow a similar proof.

Proposition 2.1. *Let N be a finite von Neumann algebra. Let A, H be groups with A non-trivial amenable and H non-amenable. Let $Q \subset N \rtimes A \wr H = M$ be an inclusion of von Neumann algebras. Assume Q is not amenable over N inside M then $Q' \cap \tilde{M}^\omega \subseteq M^\omega$.*

Proof. As mentioned above this proof follows closely the proof of Lemma 4.2 in [CPS11] as well as Lemma 5.1 in [Po06a] and other similar results in the literature.

We will prove the contrapositive so let us assume that $Q' \cap \tilde{M}^\omega \nsubseteq M^\omega$ Then proceeding as in Lemma 5.1 in [Po06a] We see that

$$L^2(Q) \prec L^2(\tilde{M}) \ominus L^2(M)$$

as Q-bimodules. Now we decompose $L^2(\tilde{M}) \ominus L^2(M)$ as an M-bimodule.

One can see that the above M-bimodule can be written as a direct sum of M-bimodules $\overline{M\tilde{\eta}_s M}^{\|\cdot\|_2}$, where the cyclic vectors $\tilde{\eta}_s$ correspond to an enumeration of all elements of $\tilde{A}^H$ whose non-trivial coordinates start and end with non-zero powers of u.

Next, for every s, we denote by η_s the element of A^H that remains from $\tilde{\eta}_s$ after deleting all nontrivial powers of u. Also for every s let $\Delta_s \subset H$ be the support of $\tilde{\eta}_s$ and observe that if $Stab_H(\tilde{\eta}_s)$ denotes the stabilizing group of $\tilde{\eta}_s$ inside H then we have $Stab_H\ (\tilde{\eta}_s)(H \setminus \Delta_s) \subset H \setminus \Delta_s$.

Hence we can consider the von Neumann algebra $K_s = N \rtimes (A \wr_{H\setminus\Delta_s} Stab_H\ (\tilde{\eta}_s))$ and using similar computations as in Lemma 5.1 of [Po06a], one can easily check that the map $x\tilde{\eta}_s y \to x\eta_s e_{K_s} y$ implements an M-bimodule isomorphism between $\overline{M\tilde{\eta}_s M}^{\|\cdot\|_2}$ and $L^2(\langle M, e_{K_s}\rangle)$.

Therefore, as M-bimodules, we have the following isomorphism

$$L^2(\tilde{M}) \ominus L^2(M) = \bigoplus L^2(\langle M, e_{K_s}\rangle).$$

Thus we can get the following weak containment of Q-bimodules

$$L^2(Q) \prec \bigoplus L^2(\langle M, e_{K_s}\rangle).$$

Notice that, since Δ_s is finite, and the action of H on itself is free, then $Stab_H(\tilde{\eta}_s)$ is finite for all s. Also, since A is an amenable group we have that $K_s \lessdot_N N$ for all s. Thus for all s we have the following weak containment of K_s-bimodules

$$L^2(K_s) \prec \bigoplus L^2(\langle K_s, e_N\rangle) \simeq \bigoplus L^2(K_s) \otimes_N L^2(K_s)$$

Now if we induce to M-bimodules and restrict to Q-bimodules and use continuity of weak containment under induction and restriction we get the following inclusions of Q-bimodules:

$$\begin{aligned} L^2(Q) \quad &\prec \quad \bigoplus L^2(\langle M, e_{K_s}\rangle) \\ &\simeq \quad \bigoplus L^2(M) \otimes_{K_s} L^2(K_s) \otimes_{K_s} L^2(M) \\ &\prec \quad \bigoplus L^2(M) \otimes_{K_s} L^2(K_s) \otimes_N L^2(K_s) \otimes_{K_s} L^2(M) \\ &\simeq \quad \bigoplus L^2(M) \otimes_N L^2(M) \\ &\simeq \quad \bigoplus L^2(\langle M, e_N\rangle) \end{aligned}$$

Thus $Q \lessdot_M N$

□

We finish this section with a final theorem which allows us to locate regular subfactors with large commutant.

Theorem 2.2. *Let N be a finite von Neumann algebra. Let A and H be groups with A non-trivial amenable and H non-amenable. Let $Q \subset N \rtimes A \wr H = M$ be a subalgebra that is not amenable over N. Let $P = Q' \cap M$. If P is a regular subfactor of M then $P \prec_M N$.*

Proof. Applying Proposition 2.1 and following the proof of Theorem 4.1 in [CPS11] we see that the deformation θ_t converges uniformly on the unit ball of P, and thus by Theorem 3.1 in [CPS11] we have that $P \prec_M N \rtimes A^H$ or $P \prec_M N \rtimes H$.

Following the same argument as Theorem 4.1 [CPS11] if we assume that $P \prec_M N \rtimes A^H$ and $P \not\prec_M N$ then we get $Q \prec_M N \rtimes A \wr H_0$ for some finite subgroup $H_0 \subset H$. Since A is amenable and H_0 is finite then $N \rtimes A \wr H_0 \lessdot_M N$. So since $Q \prec_M N \rtimes A \wr H_0$ then by Proposition 1.3 we have $Q \lessdot_M N \rtimes A \wr H_0$. Then by part 3 of Proposition 2.4 in [OP07] we have that $Q \lessdot_M N$ contradicting our assumption.

Thus $P \prec_M N \rtimes H$. Therefore, by Theorem 1.1, there exists nonzero projections $p \in P, q \in N \rtimes H$, a nonzero partial isometry $v \in M$, and a *-homomorphism $\varphi : pPp \to q(N \rtimes H)q$ such that $vx = \varphi(x)v, \forall x \in pPp$. Furthermore we have that $v^*v = p$ and $vv^* = \hat{q} \in \varphi(pPp)' \cap qMq$. Also, by Lemma 3.5 in [Po03] we know that pPp is a regular subalgebra of pMp.

Then for all $u \in \mathcal{N}_{pMp}(pPp)$ let us calculate:

$$\begin{aligned} \varphi(x)vuv^* &= vxuv^* \\ &= vu(u^*xu)v^* \\ &= vuv^*v(u^*xu)v^* \\ &= vuv^*\varphi(u^*xu)vv^* \\ &= vuv^*\varphi(u^*xu) \end{aligned}$$

Now assume that $P \not\prec_M N$, then by part (2) of Lemma 2.4 in [CPS11] we have that $vuv^* \in N \rtimes H$. Since pPp is regular in pMp we would then get that $M \prec_M N \rtimes H$. However, this is impossible since the fact that A is nontrivial implies that $[M : N \rtimes H] = \infty$.

□

3. Proof of Main Theorems

In this section we prove our main theorem. Our main technical tool is the following, which is proposition 2.7 in [PV11]. Before we state the result let us recall that two von Neumann subalgebras $M_1, M_2 \subset M$ of a finite von Neumann algebra M are said to form a commuting square if $E_{M_1}E_{M_2} = E_{M_2}E_{M_1}$.

Theorem 3.1 (Popa-Vaes, [PV11])**.** *Let (M, τ) be a tracial von Neumann algebra with von Neumann subalgebras $M_1, M_2 \subset M$. Assume that M_1 and M_2 form a commuting square and that M_1 is regular in M. If a von Neumann subalgebra $Q \subset pMp$ is amenable relative to both M_1 and M_2, then Q is amenable relative to $M_1 \cap M_2$.*

Notice that this theorem allows us to eliminate the case where Q is amenable over M_1. More specifically we have the following observation.

Proposition 3.2. *Let G_1 and G_2 be groups. Let A be a finite amenable von Neumann algebra with an action of $G_1 \times G_2$, and let $Q \subset A \rtimes G_1 \times G_2$ be a nonamenable subalgebra. Then there exists an i such that Q is not amenable over $A \rtimes G_i$.*

Proof. If we let $A \rtimes G_i = M_i$ then it is easy to see that $M_1, M_2 \subset M$ form a commuting square. So if Q is amenable over both M_i we would have that it would be amenable over the intersection, which is A. Since A is amenable this would imply that Q is amenable. □

Finally combining the above results we can prove our main theorem (Theorem 0.1).

Proof. First let us mention that for the case $n = 1$, this is equivalent to the primeness of II_1-factors arising from Bernoulli shifts, which was proven in [Po06a].

Now notice that we can write M as $M = N_i \rtimes_\sigma A_i \wr H_i$, where $N_i = L(A_1 \wr H_1)\overline{\otimes} \dots \overline{\otimes} L(A_{i-1} \wr H_{i-1})\overline{\otimes} L(A_{i+1} \wr H_{i+1})\overline{\otimes} \dots \overline{\otimes} L(A_n \wr H_n)$ and σ is the trivial action.

Let us define $\widehat{Q_i} = (Q_i)' \cap M = Q_1\overline{\otimes} \dots \overline{\otimes} Q_{i-1}\overline{\otimes} Q_{i+1}\overline{\otimes} \dots \overline{\otimes} Q_k$. Since $H_i \wr \Gamma_i$ does not have property Gamma for all i this implies, in particular, that Q_1 is non-amenable. By proposition 3.2, where we let $A = \mathbb{C}$, we know that there is an i such that Q_1 is not amenable over N_i

Since $\widehat{Q_1}$ is a regular subalgebra of M, then by Theorem 2.2 we get that $\widehat{Q_1} \prec_M N$.

We complete the argument by following Proposition 12 and the induction argument of the proof of Theorem 1 in [OP03].

□

Before we prove our final theorem let us recall the following definition:

Definition 3.3. We say that two group Γ and Λ are *measure equivalent*, $\Gamma \simeq_{ME} \Lambda$ is there is a diffuse abelian von Neumann algebra, A, and free ergodic trace preserving actions, σ, ρ of Γ and Λ, respectively, such that $A \rtimes_\sigma \Gamma \simeq (A \rtimes_\rho \Lambda)^t$, and the isomorphism takes A onto A^t.

With this definition we can now prove our final result (Corollary 0.2.)

Proof. Let $A_1 \wr H_1, ..., A_n \wr H_n$ be as above, and let $K_1, ..., K_m$ be groups. Since $A_1 \wr H_1 \times \cdots \times A_n \wr H_n \simeq_{ME} K_1 \times \cdots \times K_m$ and $A_i \wr H_i$ is nonamenable for all i, then K_j is nonamenable for all j.

Now we know that there are actions on $L^\infty(X)$ such that $M = L^\infty(X) \rtimes A_1 \wr H_1 \times \cdots \times A_n \wr H_n \simeq (L^\infty(X) \rtimes K_1 \times \cdots \times K_m)^t$. We may assume that $t = 1$.

Let $N_i = A \rtimes A_1 \wr H_1 \times \dots A_{i-1} \wr H_{i-1} \times A_{i+1} \wr H_{i+1} \times \cdots \times A_n \wr H_n$, so that we have $M = N_i \rtimes A_i \wr H_i$. As in the proof of the previous theorem, since K_i is nonamenable, there is an i such that $L(K_1)$ is nonamenable over N_i. Now by the proof of Theorem 2.2 this implies that $L(K_1)' \cap M = L(K_2 \times \cdots \times K_m) \prec N_i \rtimes H_i$. Thus by Lemma 2.2 in [CPS11] we have that $A \rtimes K_2 \times \cdots \times K_m \prec N_i \rtimes H_i$. Now since $A \rtimes K_2 \times \cdots \times K_m$ is a regular subalgebra we have by Theorem 2.2 that $A \rtimes K_2 \times \cdots \times K_m \prec N_i$.

Notice that now we can follow exactly as in the proof of Corolarry C in [CS10] to get our desired result.

$\square$

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J. Owen Sizemore, UCLA, Math Sciences Building, Los Angeles, CA 90095-1555
E-mail address: sizemore@math.ucla.edu

Adam Winchester, UCLA, Math Sciences Building, Los Angeles, CA 90095-1555
E-mail address: lagwadam@math.ucla.edu